Fuzzification

By considering quantities as uncertain:

Imprecision

Ambiguity

Vagueness

Intuition

Using our intelligence and understanding.

Intuition involves contextual and semantic knowledge about an issue. It can also involve linguistic truth-values about the knowledge.

Note: they are overlapping.

Inference

Using knowledge to perform deductive reasoning.

Example:

Let U be a universe of triangles.

$$U = \{ (A\ B\ C) \mid A \geq B \geq C \geq 0 \quad A + B + C = 180^\circ \}$$

We can define the following 5 types of triangles:

I: Approximate isosceles triangle

R: Approximate right triangle

IR: Approximate isosceles and right triangle

E: Approximate equilateral triangle

T: Other triangles

Inference

$\mu_I(A\ B\ C) = 1 - 1/60° \min(A - B, B - C)$

$\mu_R(A\ B\ C) = 1 - 1/90° \ |A - 90°|$

$IR = I \cap R$

$\mu_{IR}(A\ B\ C) = \min [\ \mu_I(A\ B\ C), \mu_R(A\ B\ C)\ ]$

$= 1 - \max[1/60° \min(A - B, B - C), 1/90° \ |A - 90°|]$

$\mu_E(A\ B\ C) = 1 - 1/180° \ (A - C)$

$T = (I \cup R \cup E)' = I' \cap R' \cap E'$

$= \min\{1 - \mu_I, 1 -1\ \mu_R, 1 - \mu_E\}$

$= 1/180° \min\{3(A - B), 3(B - C), 2|\ A - 90°|, A - C\}$

Rank Ordering

Assessing preference by a single individual, a pole, a committee, and other opinion methods can be used to assign membership values to a fuzzy variable.

Preference is determined by pair wise comparisons which determine the order of memberships.

Angular Fuzzy Sets

Angular Fuzzy sets are defined on a universe of angles with 2π as cycle.

The linguistic values vary with θ and their memberships are

$$\mu_t(\theta) = t \bullet \tan(\theta)$$

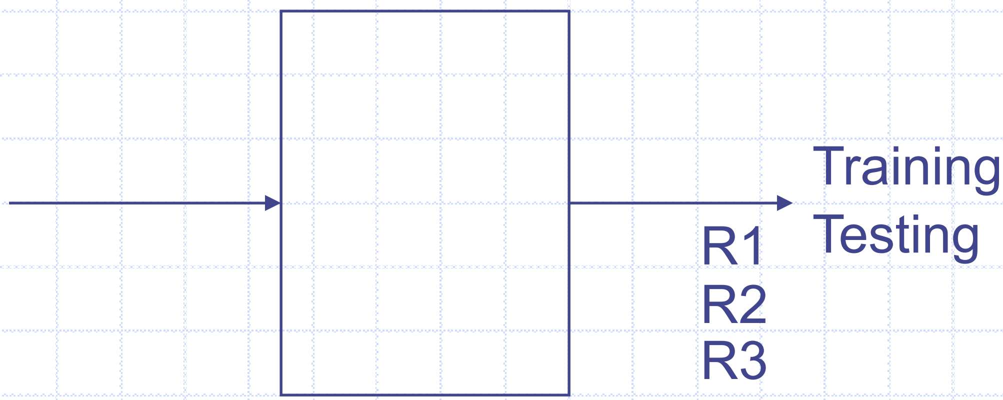
Angular Fuzzy sets are useful for situations:

Having a natural basis in polar coordinates, or the variable is cyclic.

Neural Networks

We have the data sets for inputs and outputs, the relationship between I/O may be highly nonlinear or not known.

We can classify them into different fuzzy classes.



Then, the output may not only be 0 or 1!

Neural Networks

R1	0	0.2	
R2	0	0.7	memberships
R3	0	0.1	

Once the neural network is trained and tested, it can be used to find the membership of any other data points in the fuzzy classes (# of outputs)

Genetic Algorithms

Crossover

Mutation random selection

Reproduction

Chromosomes

Fitness Function

Stop (terminate conditions)

  Converge
  Reach the #limit

Inductive Reasoning

Deriving a general consensus from the particular (from specific to generic)

The induction is performed by the entropy minimization principle, which clusters most optimally the parameters corresponding to the output classes.

The method can be useful for complete systems where the data are abundant and static.

The intent of induction is to discover a law having objective validity and universal application.

Inductive Reasoning

Particular → General

Maximize entropy
Computing mean probability
Minimize entropy

The entropy is the expected value of information.

Many entropy definitions!

A survey paper

Inductive Reasoning

One example:

$S(x) = p(x)\ S_p(x) + q(x)\ S_q(x)$

$S_p(x) = -[P_1(x)\ \ln(P_1(x)) + P_2(x)\ \ln(P_2(x))]$

$S_q(x) = -[q_1(x)\ \ln(q_1(x)) + q_2(x)\ \ln(q_2(x))]$

$$P_k(x) = \frac{n_k(x) + 1}{n(x) + 1}$$

$$q_k(x) = \frac{N_k(x) + 1}{N(x) + 1}$$

$$P(x) = \frac{n(x)}{n}$$

Inductive Reasoning

Where:

$n_k(x)$: # of class k samples in [x1,x1+x]

n(x): Total # of samples in [x1,x1+x]

$N_k(x)$: # of class k samples in [x1+x,x2]

N(x): Total # of classes in [x1+x,x2]

n = Total # of samples in [x1,x2]

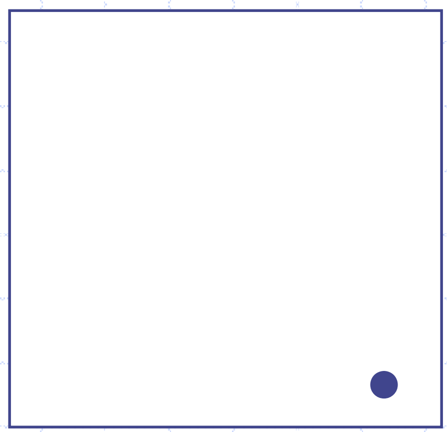
Move x in [x1,x2], and compute the entropy for each x to find the maximum / minimum entropy.

Note: there are many approaches to compute entropy.

Defuzzification
(Fuzzy-To-Crisp conversions)

Using fuzzy to reason, to model
Using crisp to act
Like analog → digital → analog

Defuzzification is the process: round it off to the nearest vertex.



Fuzzy set (collection of membership values).

Defuzzification (Fuzzy-To-Crisp conversions)

A vector of values → reduce to a single scalar quantity: most typical or representative value.

Fuzzification – Analysis – Defuzzification – Action

λ-cuts for fuzzy sets
(α-cuts, some books)

A_λ, $0 \leq \lambda \leq 1$
$A_\lambda = \{x \mid \mu_A(x) \geq \lambda\}$

Note: A_λ is a crisp set derived from the original fuzzy set.

$\lambda \in [0,1]$ can have an infinite number of values. Therefore, there can be infinite number of λ-cut sets.

Defuzzification
(Fuzzy-To-Crisp conversions)

Example:

$A = \{1/a + 0.9/b + 0.6/c + 0.3/d + 0.01/e + 0/f\}$

$A_1 = \{a\}$ or $A_1 == \{1/a + 0/b + 0/c + 0/d + 0/e + 0/f\}$

$A_{0.9} = \{a,b\}$

$A_{0.3} = \{a,b,c,d\}$

$A_{0.6} = \{a,b,c\}$

$A_{0.01} = \{a,b,c,d,e\}$

$A_0 = x = \{a,b,c,d,e,f\}$

Defuzzification (Fuzzy-To-Crisp conversions)

λ-cut re-scales the memberships to 1 or 0

The properties of λ-cut:

1. $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$
2. $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$
3. $(A')_\lambda \neq (A_\lambda)'$ except for $x = 0.5$
4. $A_\alpha \subseteq A_\lambda \; \forall \; \lambda \leq \alpha$ and $0 \leq \alpha \leq 1$

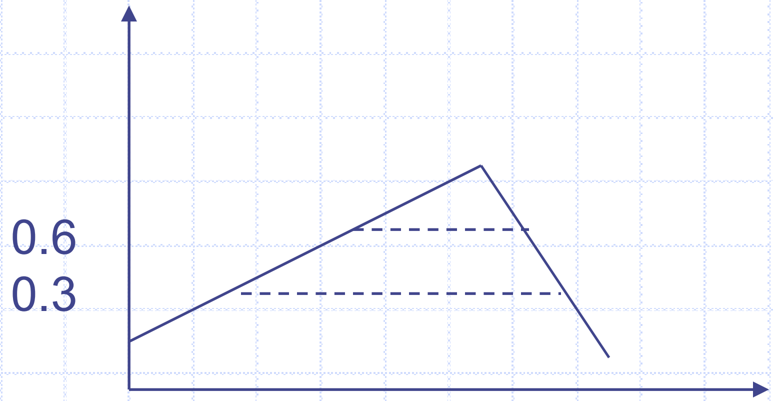
$A_0 = X$

Core $= A_1$

Support $= A_0^+$

Boundaries $= [A_0 + A_1]$

Defuzzification
(Fuzzy-To-Crisp conversions)



λ-cuts for fuzzy relations

$$R = \begin{pmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{pmatrix}$$

Defuzzification (Fuzzy-To-Crisp conversions)

- We can define λ-cut for relations similar to the one for sets

$$R_\lambda = \{(x\ y) \mid \mu_R(x\ y) \geq \lambda\}$$

$$R_1 = \begin{pmatrix} 1 & & & & \\ & 1 & & \phi & \\ & & 1 & & \\ & \phi & & 1 & \\ & & & & 1 \end{pmatrix} \qquad R_{0.9} = \begin{pmatrix} 1 & & & & \\ & 1 & & \phi & \\ & & 1 & & \\ & \phi & & 1 & \\ & & 1 & & 1 \end{pmatrix}$$

$$R_0 = E$$

Defuzzification
(Fuzzy-To-Crisp conversions)

λ-cuts on relations have the following properties:

$(R \cup S)_\lambda = R_\lambda \cup S_\lambda$

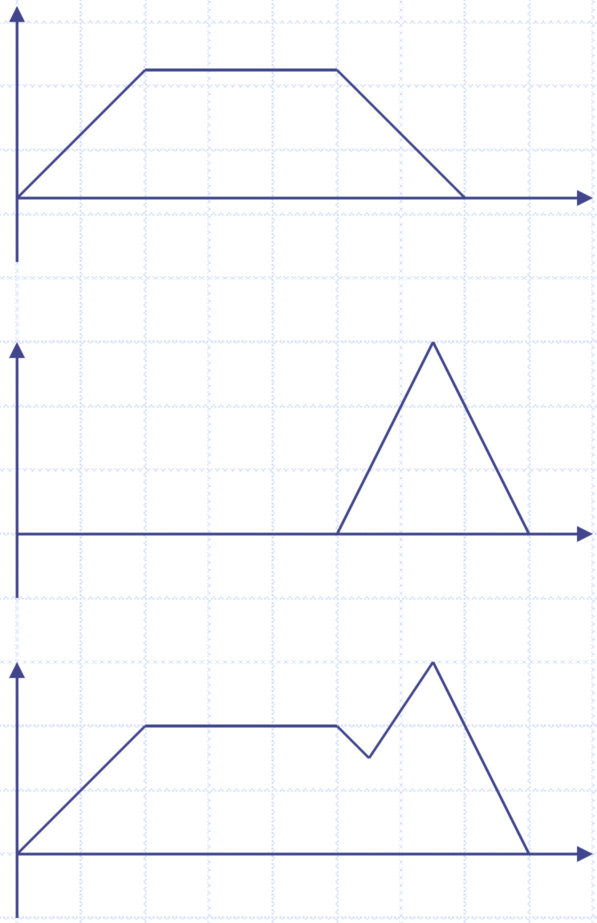
$(R \cap S)_\lambda = R_\lambda \cap S_\lambda$

$(R')_\lambda \neq (R_\lambda)'$

$R_\alpha \leq R_\lambda \ \forall \ \lambda \leq \alpha$ and $0 \leq \alpha \leq 1$

Defuzzification Methods

fuzzy set $\rightarrow$ a single scalar quantity
fuzzy quantity $\rightarrow$ precise quantity



O1

O2

$O = O1 \cup O2$

Defuzzification Methods

A fuzzy output can have many output parts

$$C = \bigcup_{i=1}^{k} C_i$$

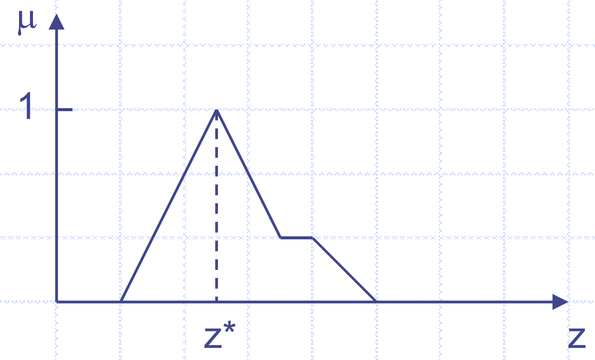
Many methods can be used for defuzzification.

They are listed in the following slides

Defuzzification Methods

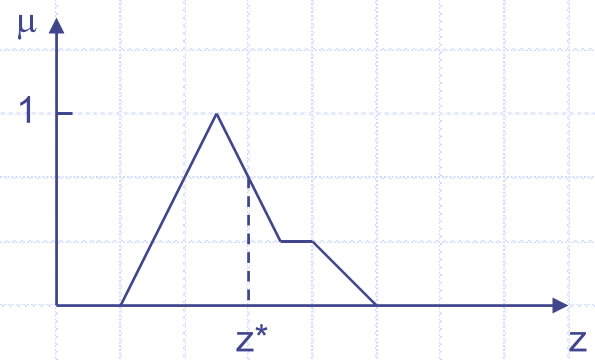
Max-membership principle

$$\mu_c(Z^*) \geq \mu_c(z)\ \forall\ z \in Z$$



Centroid principle

$$z^* = \frac{\int \mu_c(z) \bullet z dz}{\int \mu_c(z) dz}$$

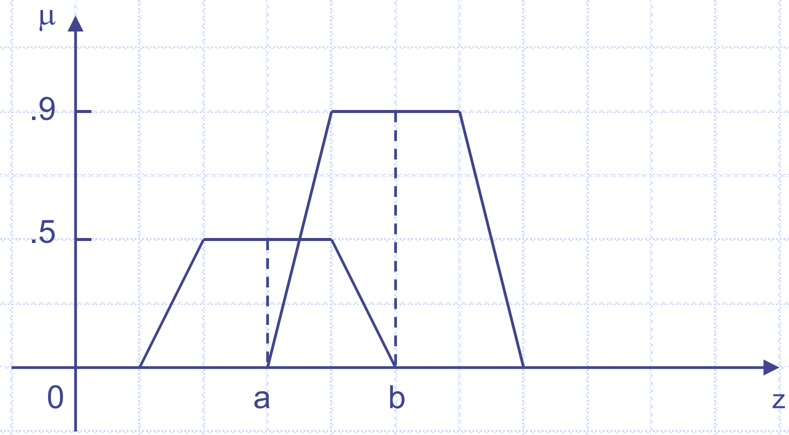


Note: It relates to moments.

Defuzzification Methods

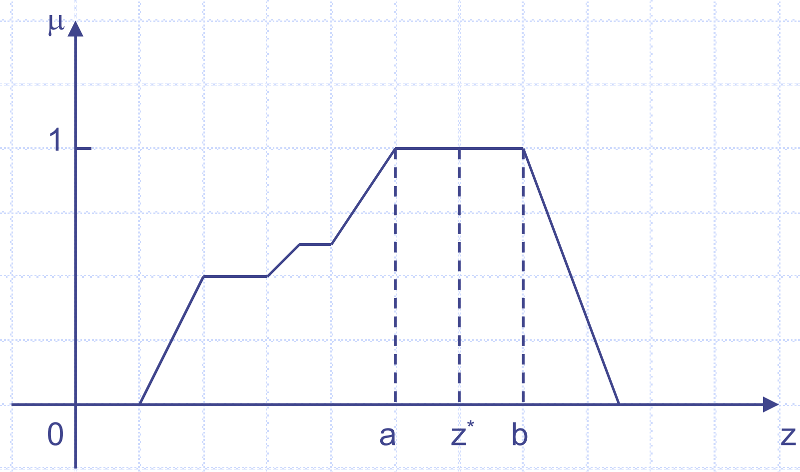
Weighted average method
(Only valid for symmetrical output membership functions)

$$z^* = \frac{\sum \mu_{\underset{\sim}{c}}(\bar{z}) \bullet \bar{z}}{\sum \mu_{\underset{\sim}{c}}(\bar{z})}$$



Mean-max membership
(middle-of-maxima method)

$$z^* = (a+b)/2$$



Defuzzification Methods

Example:

A railroad company intends to lay a new rail line in a particular part of a county. The whole area through which the new line is passing must be purchased for right-of-way considerations. It is surveyed in three stretches, and the data are collected for analysis. The surveyed data for the road are given by the sets $\underset{\sim}{B}_1, \underset{\sim}{B}_2$ and $\underset{\sim}{B}_3$, where the sets are defined on the universe of right-of-way widths, in meters. For the railroad to purchase the land, it must have an assessment of the amount of land to be bought. The three surveys on the right-of-way width are ambiguous , however, because some of the land along the proposed railway route is already public domain and will not need to be purchased. Additionally, the original surveys are so old (circa 1860) that some ambiguity exists on the boundaries and public right-of-way for old utility lines and old roads. The three fuzzy sets $\underset{\sim}{B}_1, \underset{\sim}{B}_2$ and $\underset{\sim}{B}_3$, shown in the figures below, represent the uncertainty in each survey as to the membership of the right-of-way width, in meters, in privately owned land.

We now want to aggregate these three survey results to find the single most nearly representative right-of-way width (z) to allow the railroad to make its initial estimate

Defuzzification Methods

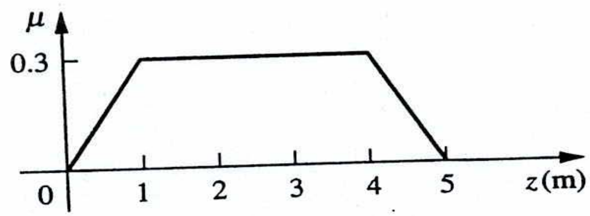


FIGURE 5.9
Fuzzy set $\underset{\sim}{B}_1$: Public right-of-way width (z) for survey 1.

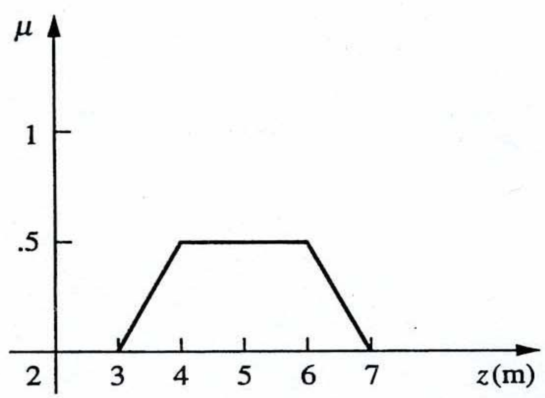


FIGURE 5.10
Fuzzy set $\underset{\sim}{B}_2$: Public right-of-way width (z) for survey 2.

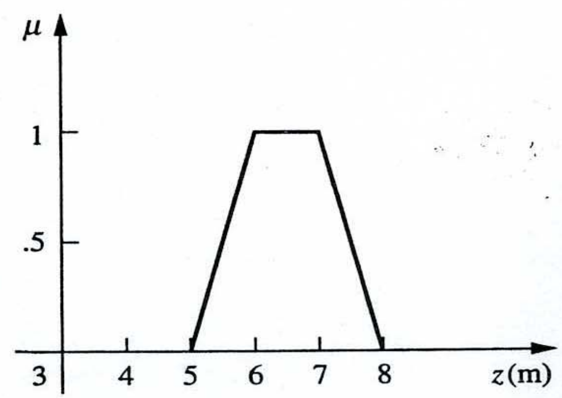


FIGURE 5.11
Fuzzy set $\underset{\sim}{B}_3$: Public right-of-way width (z) for survey 3.

Defuzzification Methods

Centroid method:

$$z^* = \frac{\int \mu_{\underset{\sim}{B}}(z) \bullet z dz}{\int \mu_{\underset{\sim}{B}}(z) dz} =$$

$$\left[\int_0^1 (.3z)zdz + \int_1^{3.6} (.3z)dz + \int_{3.6}^4 \left(\frac{z-3}{2}\right)zdz + \int_4^{5.5} (.5)zdz + \int_{5.5}^6 (z-5)zdz + \int_6^7 zdz + \int_7^8 (8-z)zdz\right]$$

$$\div \left[\int_0^1 (.3z)dz + \int_1^{3.6} (.3)dz + \int_{3.6}^4 \left(\frac{z-3}{2}\right)dz + \int_4^{5.5} (.5)dz + \int_{5.5}^6 (z-5)dz + \int_6^7 dz + \int_7^8 (8-z)dz\right]$$

$$= 4.9 meters$$

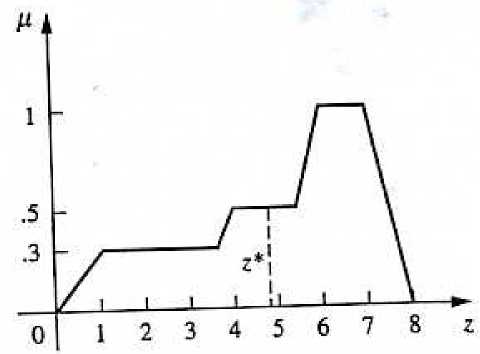


FIGURE 5.12
The centroid method for finding z^*.

Defuzzification Methods

Weighted-Average Method:

$$z^* = \frac{(.3 \times 2.5) + (.5 \times 5) + (1 \times 6.5)}{.3 + .5 + 1} = 5.41 meters$$

Mean-Max Method: $(6+7)/2 = 6.5 meters$

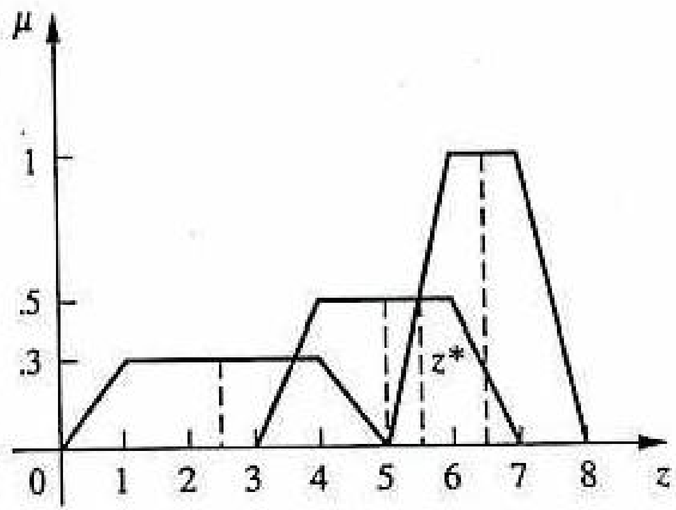


FIGURE 5.13
The weighted average method for finding z^*.

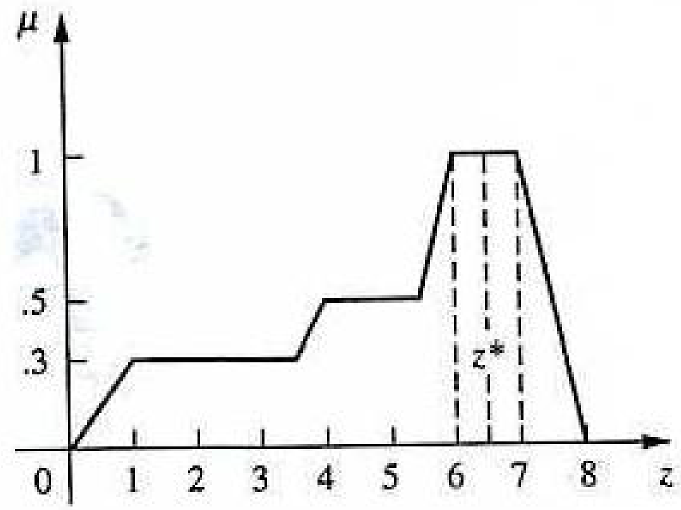


FIGURE 5.14
The mean-max membership method for finding z^*.

Defuzzification Methods

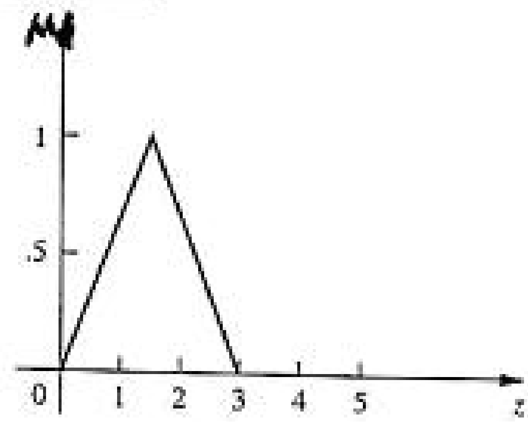


FIGURE 5.15
Membership in viscosity of oil sample 1, B_1.

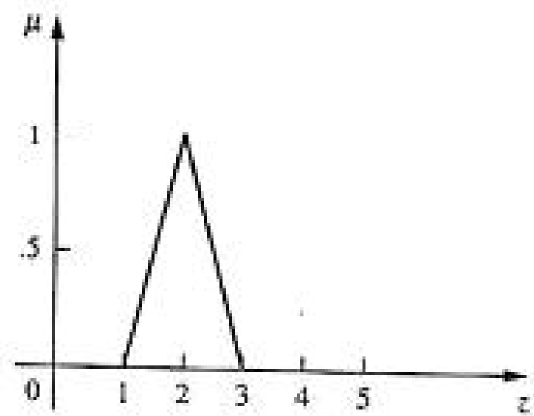


FIGURE 5.16
Membership in viscosity of oil sample 2, B_2.

Defuzzification Methods

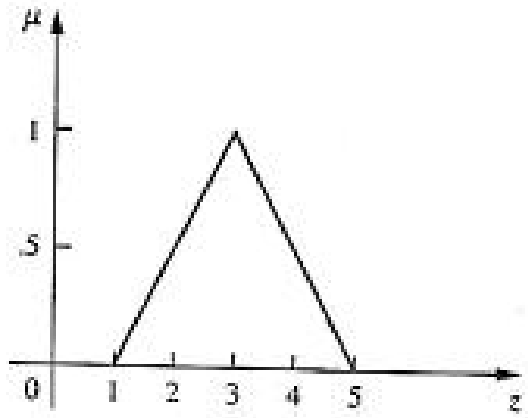


FIGURE 5.17
Membership in viscosity of oil sample 3, B_3.

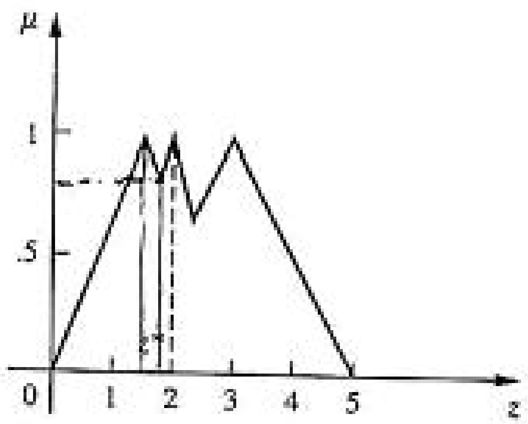


FIGURE 5.18
Logical union of three fuzzy sets B_1, B_2, and B_3.

Defuzzification Methods

According to the centroid method,

$$z^* = \frac{\int \mu_{\underset{\sim}{B}}(z) z dz}{\int \mu_{\underset{\sim}{B}}(z) dz} =$$

$$\begin{bmatrix} \int_0^{1.5} (.67z) z dz + \int_{1.5}^{1.8} (2 - .67z) z dz + \int_{1.8}^{2} (z-1) z dz + \int_2^{2.33} (3-z) z dz \\ + \int_{2.33}^{3} (.5z - .5) z dz + \int_3^5 (2.5 - .5z) z dz \end{bmatrix}$$

$$\div \begin{bmatrix} \int_0^{1.5} (.67z) dz + \int_{1.5}^{1.8} (2 - .67z) dz + \int_{1.8}^{2} (z-1) dz + \int_2^{2.33} (3-z) dz \\ + \int_{2.33}^{3} (.5z - .5) dz + \int_3^5 (2.5 - .5z) dz \end{bmatrix}$$

$$= 2.5$$

Defuzzification Methods

The centroid value obtained, z^*, is shown in the figure below:

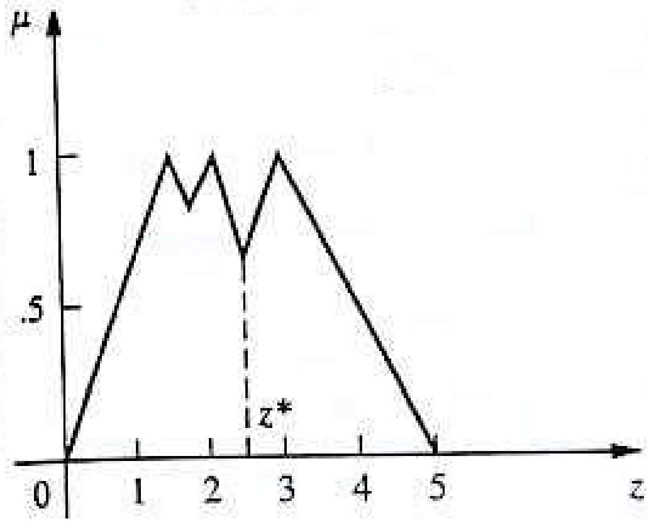


FIGURE 5.19
Centroid value z^* for three fuzzy oil samples.

Defuzzification Methods

According to the weighted average method:

$$z^* = \frac{(1\times 1.5)+(1\times 2)+(1\times 3)}{1+1+1} = 2.25$$

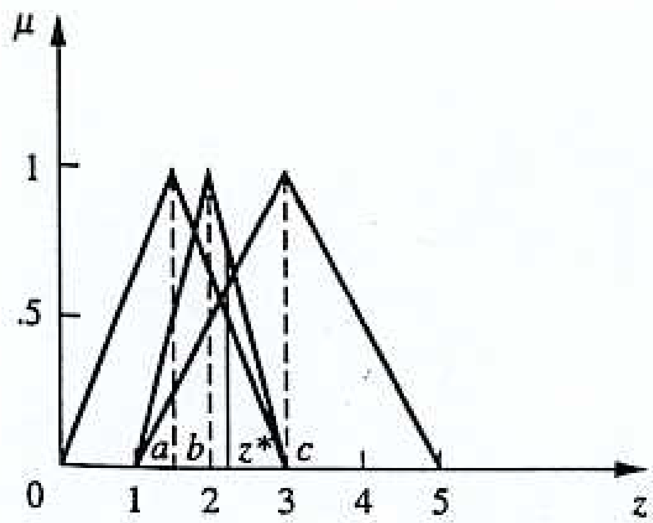


FIGURE 5.20
Weighted average method for z^*.

Defuzzification Methods

Center of sums Method

Faster than any defuzzification method

Involves algebraic sum of individual output fuzzy sets, instead of their union

Drawback: intersecting areas are added twice.

$$z^* = \frac{\int_z z \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\,dz}{\int_z \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\,dz}$$

It is similar to the weighted average method, but the weights are the areas, instead of individual membership values.

Defuzzification Methods

$z_1 = 4$ $z_2 = 8$

$$\frac{z_1 s_1 + z_2 s_2}{s_1 + s_2} = \frac{4\times 3 + 8\times 2}{3+2} = \frac{28}{5} = 5.6$$

or

$$\int_0^2 z(0.25z)dz + \int_2^6 0.5zdz + \int_6^8 (-0.25z+2)zdz$$

$$+\int_6^8 (0.5z-3)zdz + \int_8^{10}(-0.5z+5)zdz$$

$$= 0.25\frac{z^3}{3}\Big|_0^2 + 0.5\frac{z^2}{2}\Big|_2^6 + 0.25\frac{z^3}{3}\Big|_6^8$$

$$-\frac{z^2}{2}\Big|_6^8 + \frac{5}{2}z^2\Big|_8^{10} - 0.5\frac{z^3}{3}\Big|_8^{10} = 28$$

Defuzzification Methods

$$\int_0^2 (0.25z)dz + \int_2^6 0.5dz + \int_6^8 (-0.25z+2)dz$$

$$+\int_6^8 (0.5z-3)dz + \int_8^{10} (-0.5z+5)dz$$

$$= 0.25\frac{z^2}{2}\bigg|_0^2 + 0.5z\bigg|_2^6 + 0.25\frac{z^2}{2}\bigg|_6^8 - z\bigg|_6^8 - 0.5\frac{z^2}{2}\bigg|_8^{10} + 5z\bigg|_8^{10}$$

$$= 5$$

$$z^* = 28/5 = 5.6$$

Defuzzification Methods

Center of Sums Method

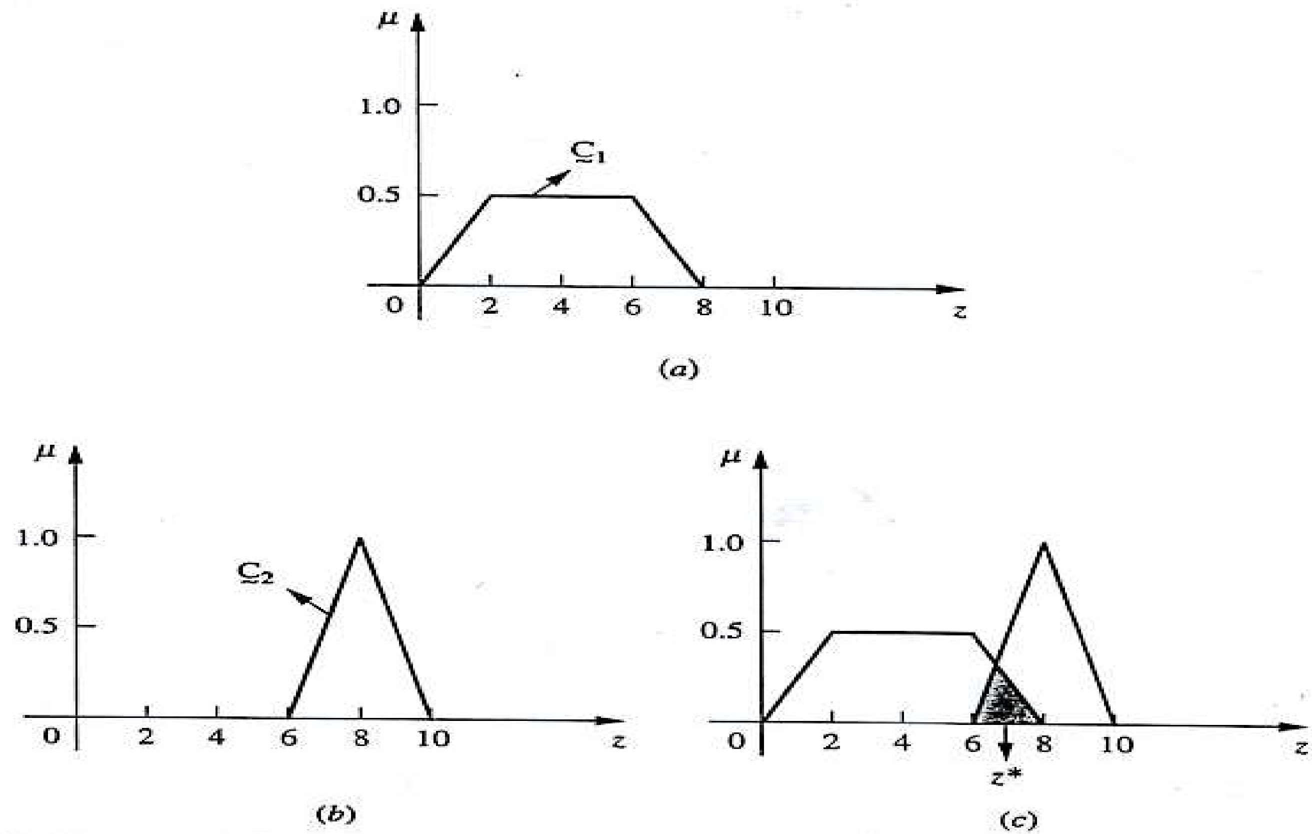


FIGURE 5.21
Center of sums method: (*a*) first membership function; (*b*) second membership function; and (*c*) defuzzification step.

Defuzzification Methods

Using Center of sums:

S1 = 0.5 * 0.5(8+4) = 3

S2 = 0.5 * 1 * 4 = 2

Center of the largest area: if output has at least two convex sub-regions

$$z^* = \frac{\int \mu_{\underset{\sim}{C}_m}(z)\,z\,dz}{\int \mu_{\underset{\sim}{C}_m}(z)\,dz}$$

Where $\underline{C}_m$ is the convex sub-region that has the largest area making up $\underline{C}_k$. (see figure)

Defuzzification Methods

- Center of sums method

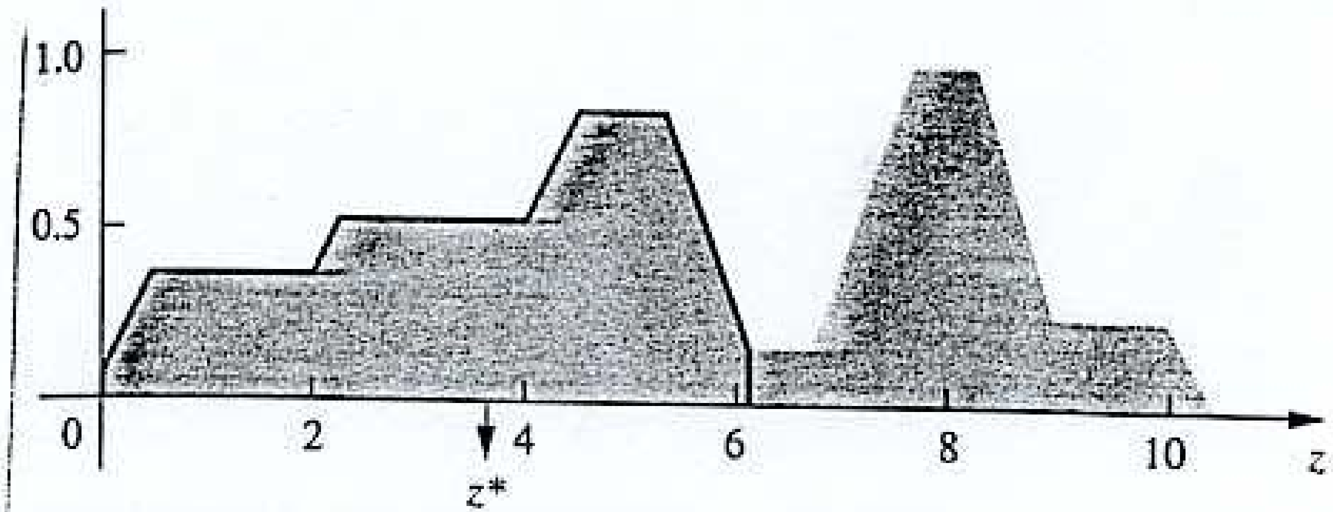


FIGURE 5.22
Center of largest area method (outlined with bold lines), shown for a nonconvex $\underset{\sim}{C}_k$.

Defuzzification Methods

First (or Last) of Maxima method

This method uses the overall output or union of all individual output fuzzy sets to determine the smallest value of the domain with maximized membership degree in each output set. The equations for z^* are as follows:

First, the largest height in the union is determined:

$$hgt\left(\underset{\sim}{C}_k\right) = \sup_{z \in Z} \mu_{\underset{\sim}{C}_k}(z)$$

Then the first of the maxima is found:

$$z^* = \inf_{z \in Z}\left\{z \in Z \mid \mu_{\underset{\sim}{C}_k}(z) = hgt\left(\underset{\sim}{C}_k\right)\right\}$$

Defuzzification Methods

First (or last) of Maxima method

- An alternative to this method is called the last of maxima, and it is given by:

$$z^* = \sup_{z \in Z} \left\{ z \in Z \mid \mu_{\underset{\sim}{C_k}}(z) = hgt\left(\underset{\sim}{C_k}\right) \right\}$$

Supremum (Sup): the least upper bound
Infimum (Inf): the greatest lower bound

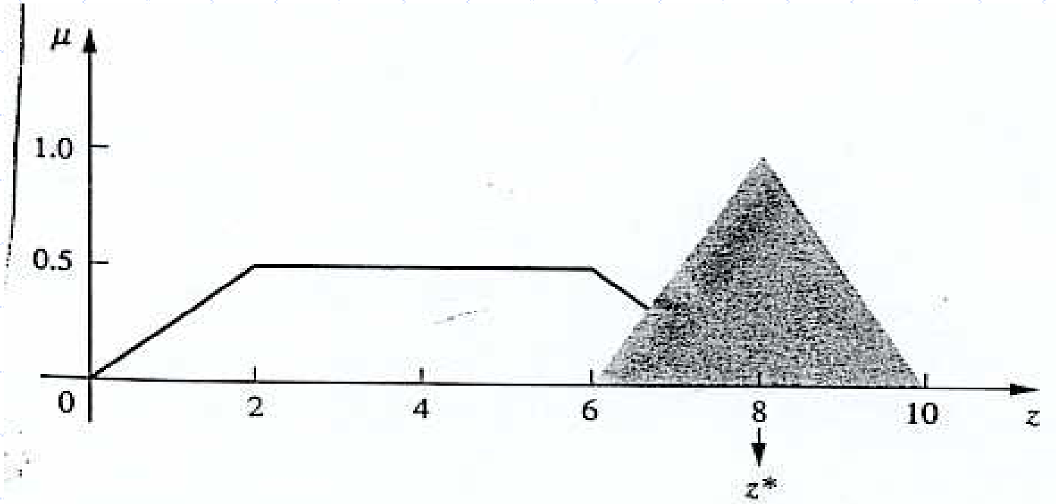


FIGURE 5.23
First of max (and last of max) method.

Defuzzification Methods

Continuation of the railroad example, the results of the different methods can be shown graphically as follows:

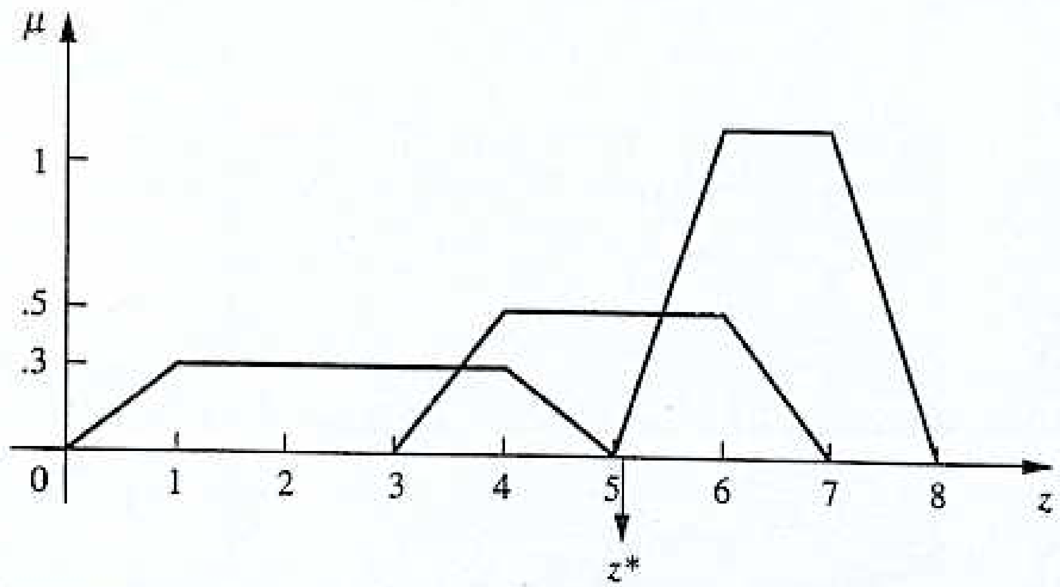


FIGURE 5.24
Center of sums result for Example 5.5.

Defuzzification Methods

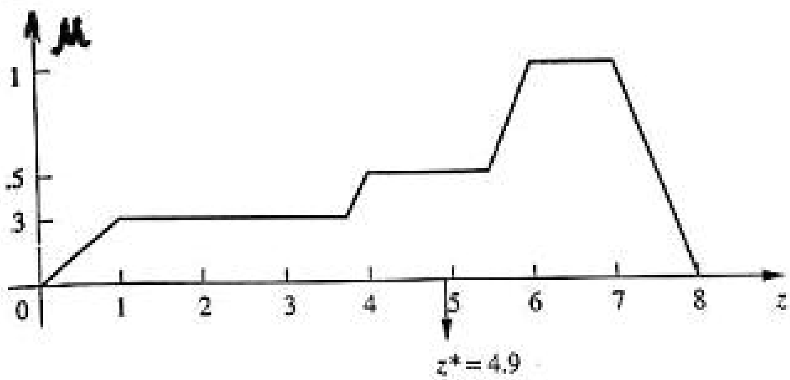


FIGURE 5.25
Output fuzzy set for Example 5.5 is convex.

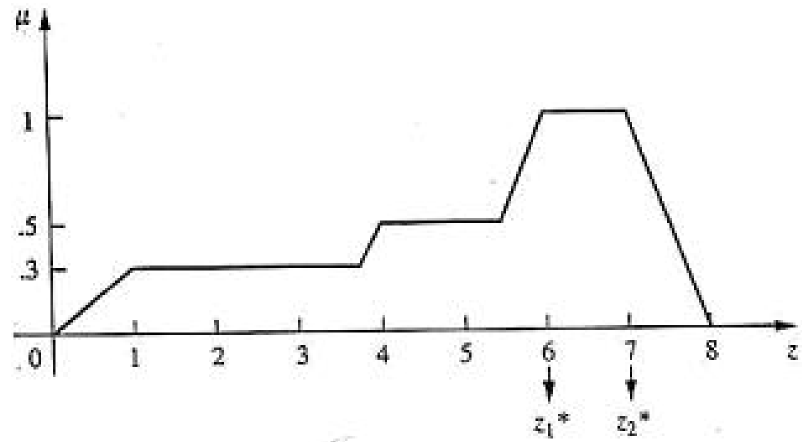


FIGURE 5.26
First of maxima solution ($z_1^* = 6$) and last of maxima solution ($z_2^* = 7$).

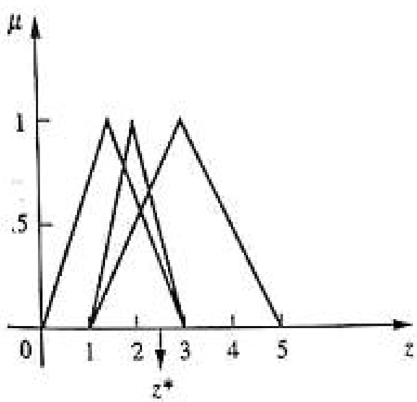


FIGURE 5.27
Center of sums solution for Example 5.6.

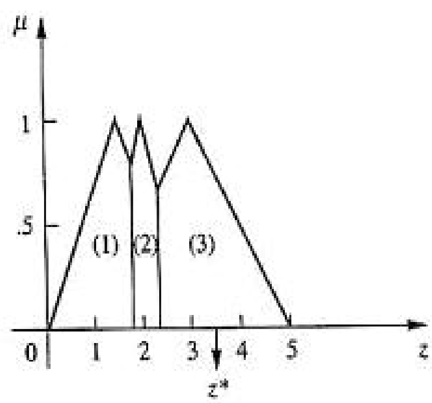


FIGURE 5.28
Center of largest area method for Example 5.6.

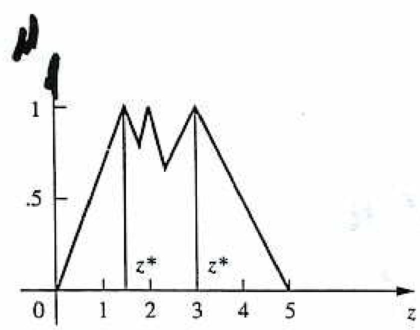
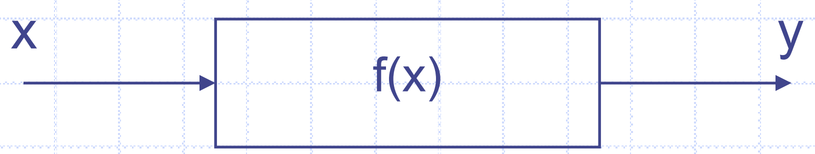


FIGURE 5.29
First of maxima gives $z^* = 1.5$ and last of maxima gives $z^* = 3$.

Fuzzy Arithmetic, Numbers, Vectors

The Extension Principle



How to find y if x is fuzzy, f is fuzzy or both are fuzzy

Crisp function, Mapping and Relation

For a set A defined on universe X, its image, set B on the universe Y is found from the mapping

$$B = f(A) = \{ y \mid \forall x \in A, y = f(x)$$

B is defined by its characteristic value

$$X_B(y) = X_{f(A)}(y) = \bigcup_{y = f(x)} X_A(x)$$

Note: $\cup$ means max

Crisp function, Mapping and Relation

Example:

$A = \{0/-2 + 0/-1 + 1/0 + 1/1 + 0/2\}$

$X = \{-2,-1,0,0,1,2\}$

If $y = |4x| + 2$ $\quad\quad$ $Y = \{2,6,10\}$

$X_B(2) = \cup \{X_A(0)\} = 1$

$X_B(6) = \cup \{X_A(-1), X_A(1)\} = \cup \{0,1\} = 1$

$X_B(10) = \cup \{X_A(-2), X_A(2)\} = \cup \{0,0\} = 0$

$B = \{1/2 + 1/6 + 0/10\}$ or $B = \{2,6\}$

Crisp function, Mapping and Relation

We may consider the universe

$X = \{-2,-1,0,1,2\}$ and universe $Y = \{0,1,2,\ldots,9,10\}$

The relation describing this mapping

$$R = \begin{array}{c|ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline -2 & & & & & & & & & & & 1 \\ -1 & & & & & & & 1 & & & & \\ 0 & \phi & & & 1 & & & & & & \phi & \\ 1 & & & & & & & 1 & & & & \\ 2 & & & & & & & & & & & 1 \end{array}$$

Crisp function, Mapping and Relation

If $A = \{0/-2 + 0/-1 + 1/0 + 1/1 + 0/2\}$

Then, $B = A \circ R$

$$X_B(y) = \bigcup_{x \in X} (X_A(A) \cap X_R(x\ y))$$

$$= \begin{cases} 1 & \text{for } y = 2,6 \\ 0 & \text{otherwise} \end{cases}$$

or

$B = \{0/0 + 0/1 + 1/2 + 0/3 + 0/4 + 0/5 + 1/6 + 0/7 + 0/8 + 0/9 + 0/10\}$

Function of Fuzzy Sets – Extension Principle

B = f(A)

If A is fuzzy, B is also fuzzy.

$$\mu_B(y) = \bigcup_{f(x) = y} \mu_A(x)$$

Fuzzy Vectors:

$$\vec{a} = \{\mu_A(x_1), \ldots \mu_A(x_n)\} = \{\mu_A(x_i)\}$$
$$i = 1, 2, 3, \ldots, n$$

$$\vec{b} = \{\mu_B(y_1), \ldots \mu_B(y_n)\} = \{\mu_B(y_j)\}$$
$$j = 1, 2, 3, \ldots, m$$

$$B = A \circ R$$
$$or$$
$$\vec{b} = \vec{a} \circ R$$

Function of Fuzzy Sets – Extension Principle

General case

$$f : P\left(x_1 \times x_2 \times \ldots \times x_n\right) \to P\left(Y\right)$$

Let A1,A2,…An be defined on X1,X2,…,Xn

Then B = f(A1,A2,…,An)

$$\mu_B\left(y\right) = \max\left\{\min\left[\mu_{A1}\left(y_1\right), \mu_{A2}\left(y_2\right), \ldots, \mu_{An}\left(y_n\right)\right]\right\}$$
$$y = f\left(x_1, x_2, \ldots, x_n\right)$$

This is called Zadeh's extension principle.

Fuzzy Transform (Mapping)

Extending fuzziness in an input set to an output set.

I: fuzzy O: fuzzy f: crisp f:A → B

If x ∈ X then $\underset{\sim}{B} = \underset{\sim}{f}(x)$ is called fuzzy mapping, ~ indicates fuzzy.

It can be described as a fuzzy relation.

$$\underset{\sim}{R} = \begin{array}{c} \\ x1 \\ x2 \\ : \\ xn \end{array} \begin{array}{c} \begin{array}{ccccc} y1 & y2 & \cdots & ym \end{array} \\ \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ : & : & : & : \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{pmatrix} \end{array}$$

Fuzzy Transform (Mapping)

$$B_i = f(x_i)$$

$$\mu_{Bi}(y_i) = r_{ij}$$

$$or$$

$$\vec{b}_i = \{r_{i1}, r_{i2}, \ldots, r_{im}\}$$

More general:

$$\underset{\sim}{B} = \underset{\sim}{f}\left(\underset{\sim}{A}\right)$$

$$\mu_{\underset{\sim}{B}}(y) = \bigcup_{x \in X}\left(\mu_{\underset{\sim}{A}}(x) \cap \mu_{\underset{\sim}{R}}(xy)\right)$$

$$or$$

$$\vec{b} = a \circ R$$

$$\underset{\sim}{b}_j = \max_i \{\min(a_i, r_{ij})\}$$

$\underset{\sim}{b}_j$ is the j[th] element of the fuzzy image B

Fuzzy Transform (Mapping)

$$R = \begin{array}{cc} \begin{pmatrix} 1 & 0.8 & 0.2 & 0.1 & 0 \\ 0.8 & 1 & 0.8 & 0.2 & 0.1 \\ 0.2 & 0.8 & 1 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.8 & 1 & 0.8 \\ 0 & 0.1 & 0.2 & 0.8 & 1 \end{pmatrix} & \begin{array}{c} 40 \\ 50 \\ 60 \\ 70 \\ 80 \end{array} \end{array}$$

$$\underset{\sim}{A} = \{0.8/40 + 1/50 + 0.6/60 + 0.2/70 + 0/80\}$$

or

$$\underset{\sim}{a} = \{0.8, 1, 0.6, 0.2, 0\}$$

$$\underset{\sim}{b} = a \circ R = \{0.8, 1, 0.8, 0.6, 0.2\}$$

Practical Considerations

F:u → v, u ∈U and v ∈ V

$$\underset{\sim}{A} \subset U$$

$$\underset{\sim}{A} = \{\mu_1 / u_1 + \mu_2 / u_2 + \ldots + \mu_n / u_n\}$$

Then the extension principle is

$$f\left(\underset{\sim}{A}\right) = f\left(\mu_1 / u_1 + \mu_2 / u_2 + \ldots + \mu_n / u_n\right)$$

$$= \{\mu_1 / f(u_1) + \mu_2 / f(u_2) + \ldots + \mu_n / f(u_n)\}$$

It is a mapping called one-to-one.

Practical Considerations

Example:

u = {1,2,3}

v= f(u) = 2u – 1

A = {0.6/1 + 1/2 + 0.8/3}

Then f(A) = {0.6/1 + 1/3 + 0.8/5}

$$\text{If } \underset{\sim}{A} \subseteq U1 \times U2$$

$$\text{Then}$$

$$f\left(\underset{\sim}{A}\right) = \left\{ \sum \frac{\min[\mu_1(i), \mu_2(j)]}{f(i, j)} \middle| i \in U1, j \in U2 \right\}$$

Where $\mu_1(i)$ and $\mu_2(j)$ are the separable membership projections of μ(I,j) from U1× U2, when μ(I,j) cannot be determined.

Practical Considerations

Example:

U1 = U2 = {1,2,…,10}

A = 2 = “Approximately 2” = {0.6/1 + 1/2 + 0.8/3}

B = 6 = “Approximately 6” = {0.8/5 + 1/6 + 0.7/7}

$$= \left\{ \begin{array}{l} \frac{\min(0.6,0.8)}{5} + \frac{\min(0.6,1)}{6} + \frac{\min(0.6,0.7)}{7} \\ + \frac{\min(1,0.8)}{10} + \frac{\min(1,1)}{12} + \frac{\min(1,7)}{14} + \frac{\min(0.8,0.8)}{15} \\ + \frac{\min(0.8,1)}{18} + \frac{\min(0.8,0.7)}{21} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 0.6/5 + 0.6/6 + 0.6/7 + 0.8/10 + 1/12 \\ + 0.7/14 + 0.8/15 + 0.8/18 + 0.7/2 \end{array} \right\}$$

This mapping is unique. If not, we have to perform maximum operation!

Practical Considerations

$$\mu_{\underset{\sim}{A}}(u_1,u_2)=\max_{u=f(u_1,u_2)}\{\min[\mu_1(u_1),\mu_2(u_2)]\}$$

Example:

A = {0.2/1 + 1/2 + 0.7/4}

B = {0.5/1 + 1/2}

$$f(A,B)=A\times B$$

$$=\left\{\begin{array}{l}\dfrac{\min(0.2,0.5)}{1}+\dfrac{\max[\min(0.2,1),\min(0.5,1)]}{2}\\+\dfrac{\max[\min(0.7,0.5),\min(1,1)]}{4}+\dfrac{\min(0.7,1)}{8}\end{array}\right\}$$

$$=\{0.2/1+0.5/2+1/4+0.7/8\}$$

Practical Considerations

Example:

We want to map ordered pairs from the input universe X1 = {a,b} and X2 = {1,2,3} to an output universe Y={x,y,z}, for instance

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} x & z & x \\ x & y & z \end{pmatrix} \end{array}$$

Crisp

$$if\ \underset{\sim}{A} = \{0.6/a + 1/b\}$$

$$\underset{\sim}{B} = \{0.2/1 + 0.8/2 + 0.4/3\}$$

$$\underset{\sim}{C} = f\left(\underset{\sim}{A}, \underset{\sim}{B}\right)$$

f is described by R.

Practical Considerations

$$\mu_{\underset{\sim}{C}}(x) = \max[\min(0.2, 0.6), \min(0.4, 0.6), \min(0.2, 1)] = 0.4$$

$$\mu_{\underset{\sim}{C}}(y) = \min(1, 0.8) = 0.8$$

$$\mu_{\underset{\sim}{C}}(z) = \max[\min(0.6, 0.8), \min(1, 0.4)] = 0.6$$

$$\underset{\sim}{C} = \{0.4/x + 0.8/y + 0.6/z\}$$

Note:

$$\begin{pmatrix} 0.6 \\ 1 \end{pmatrix}(0.2, 0.8, 0.4) = \begin{pmatrix} \min(0.6, 0.2), \min(0.6, 0.8), \min(0.6, 0.4) \\ \min(1, 0.2), \min(1, 0.8), \min(1, 0.4) \end{pmatrix}$$

Practical Considerations

Consider:

$$\underset{\sim}{x} = \cos\left(\underset{\sim}{w}t\right)$$

$$\mu_{\underset{\sim}{x}}(x) = \bigcup_{x=\cos(wt)} \left[\mu_{\underset{\sim}{w}}(w)\right]$$

For t = 0, all values of $\underset{\sim}{w}$ map into a single point.

$$\underset{\sim}{w}t = 0 \rightarrow x = 1$$

$$\therefore \mu_{\underset{\sim}{x}}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

If t ≠ 0, but small, the supp $\underset{\sim}{w}$ (support of $\underset{\sim}{w}$)

The membership value of in this interval is determined in a one-to-one mapping